

①(a) Solve $y'' + 3y' + 2y = 6$.

Step 1: Solve homogeneous equation $y'' + 3y' + 2y = 0$.

The characteristic equation is $r^2 + 3r + 2 = 0$
which has roots:

$$r = \frac{-3 \pm \sqrt{3^2 - 4(1)(2)}}{2(1)} = \frac{-3 \pm \sqrt{1}}{2} = \frac{-3+1}{2}, \frac{-3-1}{2}$$
$$= -1, -2$$

Thus, $y_h = c_1 e^{-x} + c_2 e^{-2x}$

Step 2: Find a particular solution y_p to $y'' + 3y' + 2y = 6$.

Guess $y_p = A$.

Then, $y_p' = 0$

$y_p'' = 0$.

Plug these into $y'' + 3y' + 2y = 6$ to get

$$0 + 3(0) + 2A = 6.$$

So $A = 3$.

Thus, $y_p = 3$

Step 3: The general solution is

$$y = y_h + y_p = c_1 e^{-x} + c_2 e^{-2x} + 3$$

①(b) Solve $y'' - 10y' + 25y = 30x + 3$

Step 1: Solve homogeneous eqn. $y'' - 10y' + 25y = 0$.

The characteristic equation is $r^2 - 10r + 25 = 0$
which has roots

$$r = \frac{10 \pm \sqrt{(-10)^2 - 4(1)(25)}}{2(1)} = \frac{10 \pm \sqrt{0}}{2} = 5$$

Thus,

$$y_h = c_1 e^{5x} + c_2 x e^{5x}$$

Step 2: Find a particular solution y_p to

$$y'' - 10y' + 25y = 30x + 3.$$

Guess: $y_p = Ax + B$.

So, $y_p' = A$

$$y_p'' = 0$$

Plug this into $y'' - 10y' + 25y = 30x + 3$ to get

$$-10A + 25(Ax + B) = 30x + 3.$$

Regroup left-hand side to get

$$\underbrace{25Ax + (-10A + 25B)} = \underbrace{30x + 3}.$$

This gives the system

$$\begin{aligned} 25A &= 30 \\ -10A + 25B &= 3 \end{aligned}$$

Thus, $A = \frac{30}{25} = \frac{6}{5}$.

And $B = \frac{3}{25} + \frac{10}{25}A = \frac{3}{25} + \frac{10}{25} \cdot \frac{6}{5} = \frac{15}{25} = \frac{3}{5}$

So, $y_p = Ax + B = \frac{6}{5}x + \frac{3}{5}$

Step 3: The general solution to $y'' - 10y' + 25y = 30x + 3$ is thus

$$y = y_h + y_p = c_1 e^{5x} + c_2 x e^{5x} + \frac{6}{5}x + \frac{3}{5}$$

(1)(c) Solve $y'' + 4y' + 4y = 4x^2 - 8x$

Step 1: Solve the homogeneous eqn. $y'' + 4y' + 4y = 0$
which has characteristic eqn. $r^2 + 4r + 4 = 0$

The roots are

$$r = \frac{-4 \pm \sqrt{4^2 - 4(1)(4)}}{2(1)} = \frac{-4 \pm \sqrt{0}}{2} = \frac{-4}{2} = -2$$

↑
real repeated root

Thus,

$$y_h = c_1 e^{-2x} + c_2 x e^{-2x}$$

Step 2: Find a particular solution to

$$y'' + 4y' + 4y = 4x^2 - 8x$$

Guess: $y_p = Ax^2 + Bx + C$

We get $y_p' = 2Ax + B$

$$y_p'' = 2A$$

Plug this into $4y'' + 4y' + 4y = 4x^2 - 8x$ to get

$$(2A) + 4(2Ax + B) + 4(Ax^2 + Bx + C) = 4x^2 - 8x$$

Rearrange the left-hand side to get

$$4Ax^2 + (8A + 4B)x + (2A + 4B + 4C) = 4x^2 - 8x + 0$$

So we get

$$\begin{cases} 4A = 4 & \textcircled{1} \\ 8A + 4B = -8 & \textcircled{2} \\ 2A + 4B + 4C = 0 & \textcircled{3} \end{cases}$$

① gives $A = 1$.
 Plug into ② to get: $8(1) + 4B = -8 \rightarrow B = -4$
 Plug into ③ to get: $2(1) + 4(-4) + 4C = 0 \rightarrow 4C = 14$
 $\rightarrow C = \frac{14}{4} = \frac{7}{2}$

Thus,

$$y_p = Ax^2 + Bx + C = x^2 - 4x + \frac{7}{2}$$

Step 3: The general solution to

$$4y'' + 4y' + 4y = 4x^2 - 8x \text{ is}$$

$$y = y_h + y_p = c_1 e^{-2x} + c_2 x e^{-2x} + x^2 - 4x + \frac{7}{2}$$

(1)(d) Solve $y'' + 3y' - 10y = 6e^{4x}$

Step 1: Solve the homogeneous equation $y'' + 3y' - 10y = 0$ with characteristic polynomial $r^2 + 3r - 10 = 0$ which has roots:

$$r = \frac{-3 \pm \sqrt{3^2 - 4(1)(-10)}}{2(1)} = \frac{-3 \pm \sqrt{49}}{2} = \frac{-3 \pm 7}{2}$$
$$= \frac{-3+7}{2}, \frac{-3-7}{2} = 2, -5$$

Thus,

$$y_h = c_1 e^{2x} + c_2 e^{-5x}$$

Step 2: Guess a particular solution y_p to $y'' + 3y' - 10y = 6e^{4x}$.

Guess: $y_p = Ae^{4x}$
 $y_p' = 4Ae^{4x}$
 $y_p'' = 16Ae^{4x}$

Plug these into $y'' + 3y' - 10y = 6e^{4x}$ to get

$$(16Ae^{4x}) + 3(4Ae^{4x}) + 10(Ae^{4x}) = 6e^{4x}$$

This becomes $38Ae^{4x} = 6e^{4x}$

Thus, $38A = 6$.

So, $A = \frac{6}{38} = \frac{3}{19}$

So, $y_p = \frac{3}{19}e^{4x}$

Step 3: The general solution to

$y'' + 3y' - 10y = 6e^{4x}$ is

$$y = y_h + y_p = c_1 e^{2x} + c_2 e^{-5x} + \frac{3}{19} e^{4x}$$

①(e) Solve $4y'' - 4y' - 3y = \cos(2x)$

Step 1: Solve the homogeneous eqn. $4y'' - 4y' - 3y = 0$

The characteristic eqn is $4r^2 - 4r - 3 = 0$.

The roots are

$$r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(4)(-3)}}{2(4)} = \frac{4 \pm \sqrt{64}}{8} = \frac{4 \pm 8}{8}$$

$$= \frac{4+8}{8}, \frac{4-8}{8} = \frac{3}{2}, -\frac{1}{2}$$

Thus,

$$y_h = c_1 e^{\frac{3}{2}x} + c_2 e^{-\frac{1}{2}x}$$

Step 2: Find a particular solution y_p to

$$4y'' - 4y' - 3y = \cos(2x).$$

Guess: $y_p = A \cos(2x) + B \sin(2x)$

Then, $y_p' = -2A \sin(2x) + 2B \cos(2x)$

$$y_p'' = -4A \cos(2x) - 4B \sin(2x)$$

Plugging these into $4y'' - 4y' - 3y = \cos(2x)$ gives

$$-16A \cos(2x) - 16B \sin(2x) + 8A \sin(2x) - 8B \cos(2x)$$

$$-3A \cos(2x) - 3B \sin(2x) = \cos(2x).$$

Combine like terms on the left-hand side to get
$$\underbrace{(-19A - 8B)}_1 \cos(2x) + \underbrace{(8A - 19B)}_0 \sin(2x) = \cos(2x)$$

This gives

$$\begin{cases} -19A - 8B = 1 & \textcircled{1} \\ 8A - 19B = 0 & \textcircled{2} \end{cases}$$

② gives $A = \frac{19}{8}B$.

Plug this into ① to get $-19\left(\frac{19}{8}B\right) - 8B = 1$.

This gives $-\frac{361 - 64}{8}B = 1$.

$$\text{So, } B = \frac{-8}{425}$$

$$\text{Thus, } A = \frac{19}{8}B = \frac{19}{8}\left(\frac{-8}{425}\right) = \frac{-19}{425}$$

$$\text{So, } y_p = \frac{-19}{425} \cos(2x) - \frac{8}{425} \sin(2x)$$

Step 3: The general solution to $4y'' - 4y' - 3y = \cos(2x)$

is

$$y = y_h + y_p = c_1 e^{\frac{3}{2}x} + c_2 e^{-\frac{1}{2}x} - \frac{19}{425} \cos(2x) - \frac{8}{425} \sin(2x)$$

① (f) Solve $y'' + 3y = xe^{3x}$

Step 1: Solve the homogeneous eqn. $y'' + 3y = 0$.

The characteristic equation is $r^2 + 3 = 0$

Which has roots

$$r = \frac{-0 \pm \sqrt{0^2 - 4(1)(3)}}{2(1)} = \frac{\pm \sqrt{-12}}{2} = \frac{\pm \sqrt{4} \sqrt{-3}}{2}$$

$$= \frac{\pm 2\sqrt{-3}}{2} = \pm \sqrt{-3} = \pm \sqrt{3} \sqrt{-1} = \pm \sqrt{3} i$$

$0 \pm \sqrt{3} i$

Thus,

$$y_h = c_1 e^{0x} \cos(\sqrt{3}x) + c_2 e^{0x} \sin(\sqrt{3}x)$$
$$= c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)$$

Step 2: Find a particular solution y_p to

$$y'' + 3y = xe^{3x}$$

Guess: $y_p = (Ax + B)e^{3x}$

So, $y_p = Ax e^{3x} + B e^{3x}$

$$y_p' = A e^{3x} + 3Ax e^{3x} + 3B e^{3x}$$

$$y_p'' = 3A e^{3x} + 3A e^{3x} + 9Ax e^{3x} + 9B e^{3x}$$

Plugging these into $y'' + 3y = xe^{3x}$ gives

$$\underbrace{(6A+9B)e^{3x} + 9Axe^{3x}}_{y_p''} + \underbrace{3Axe^{3x} + 3Be^{3x}}_{3y_p} = xe^{3x}$$

Combining like terms gives

$$\underbrace{(6A+12B)}_0 e^{3x} + \underbrace{(12A)}_1 xe^{3x} = xe^{3x}$$

So we get

$$\begin{cases} 12A = 1 & \textcircled{1} \\ 6A + 12B = 0 & \textcircled{2} \end{cases}$$

① gives $A = \frac{1}{12}$

② gives $B = -\frac{6}{12}A = -\frac{1}{2}\left(\frac{1}{12}\right) = -\frac{1}{24}$

Thus, $y_p = (Ax+B)e^{3x} = \left(\frac{1}{12}x - \frac{1}{24}\right)e^{3x}$

Step 3: The general solution to $y'' + 3y = xe^{3x}$

is

$$y = y_h + y_p = c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x) + \left(\frac{1}{12}x - \frac{1}{24}\right)e^{3x}$$

(2)(a) Solve $y'' - y' = -3$

Step 1: Solve the homogeneous eqn $y'' - y' = 0$.

The characteristic eqn is $r^2 - r = 0$

This factors as $r(r-1) = 0$.

The roots are $r = 0, 1$.

Thus,

$$y_h = c_1 e^{0x} + c_2 e^x = c_1 + c_2 e^x$$

Step 2: Find a particular solution y_p to $y'' - y' = -3$.

We want to guess $y_p = A$.

However, the constant solution appears in y_h

So we multiply by x and instead guess

$$y_p = Ax$$

Which doesn't occur in y_h .

Then, $y_p' = A$

$$y_p'' = 0$$

Plugging this into $y'' - y' = -3$ gives

$$0 - A = -3$$

So, $A = 3$.

Thus, $y_p = 3x$

Step 3: The general solution to $y'' - y' = -3$ is

$$y = y_h + y_p = c_1 + c_2 e^x + 3x$$

②(b) Solve $y'' - 16y = 2e^{4x}$

Step 1: Solve the homogeneous eqn $y'' - 16y = 0$.

The characteristic eqn is $r^2 - 16 = 0$.

The roots are $r = \pm 4$.

Thus,

$$y_h = c_1 e^{4x} + c_2 e^{-4x}$$

Step 2: Find a particular solution y_p to

$$y'' - 16y = 2e^{4x}$$

We first think to guess $y_p = Ae^{4x}$.

However this appears as a term in y_h .

So we multiply by x to get $y_p = Axe^{4x}$

which doesn't appear in y_h .

Then, $y_p = Axe^{4x}$

$$y_p' = Ae^{4x} + 4Axe^{4x}$$

$$y_p'' = 4Ae^{4x} + 4Ae^{4x} + 16Axe^{4x}$$

plugging this into $y'' - 16y = 2e^{4x}$ gives

$$\underbrace{8Ae^{4x} + 16Axe^{4x}}_{y_p''} - \underbrace{16Axe^{4x}}_{y_p} = 2e^{4x}$$

So we get

$$8Ae^{4x} = 2e^{4x}$$

Thus, $A = \frac{1}{4}$.

So, $y_p = \frac{1}{4} x e^{4x}$.

Step 3: The general solution to $y'' - 16y = 2e^{4x}$

is

$$y = y_h + y_p = c_1 e^{4x} + c_2 e^{-4x} + \frac{1}{4} x e^{4x}$$

②(c) Solve $y'' + 2y' = 2x + 5 - e^x$

Step 1: Solve the homogeneous eqn $y'' + 2y' = 0$.

The characteristic eqn is $r^2 + 2r = 0$.

This factors as $r(r+2) = 0$.

The roots are $r = 0, -2$.

Thus,

$$y_h = c_1 e^{0x} + c_2 e^{-2x} = c_1 + c_2 e^{-2x}$$

Step 2: Find a particular solution y_p to

$$y'' + 2y' = 2x + 5 - e^x$$

Guess: $y_p = Ax^2 + Bx + Ce^x$

$$y_p' = 2Ax + B + Ce^x$$

$$y_p'' = 2A + Ce^x$$

Plugging this into $y'' + 2y' = 2x + 5 - e^x$ gives

$$\underbrace{2A + Ce^x}_{y_p''} + \underbrace{4Ax + 2B + 2Ce^x}_{2y_p'} = 2x + 5 - e^x$$

Combining like terms gives

At first you might guess $y_p = Ax + B + Ce^x$ but y_h has a constant term so bump to $Ax + B$ up by an x to get the $Ax^2 + Bx$ term

$$4Ax + (2A+2B) + 3Ce^x = 2x + 5 - e^x$$

So,

$$\begin{cases} 4A = 2 & \textcircled{1} \\ 2A + 2B = 5 & \textcircled{2} \\ 3C = -1 & \textcircled{3} \end{cases}$$

① gives $A = \frac{1}{2}$.

② gives $B = \frac{5}{2} - A = \frac{5}{2} - \frac{1}{2} = 2$

③ gives $C = -\frac{1}{3}$.

Thus, $y_p = \frac{1}{2}x^2 + 2x - \frac{1}{3}e^x$

Step 3: The general solution to

$$y'' + 2y' = 2x + 5 - e^x \text{ is}$$

$$y = y_h + y_p = c_1 + c_2 e^{-2x} + \frac{1}{2}x^2 + 2x - \frac{1}{3}e^x$$

②(c) Solve $y'' + 2y' = 2x + 5 - e^{-2x}$

Step 1: Solve the homogeneous eqn $y'' + 2y' = 0$.

The characteristic eqn is $r^2 + 2r = 0$.

This factors as $r(r+2) = 0$.

The roots are $r = 0, -2$.

Thus,

$$y_h = c_1 e^{0x} + c_2 e^{-2x} = c_1 + c_2 e^{-2x}$$

Step 2: Find a particular solution y_p to

$$y'' + 2y' = 2x + 5 - e^{-2x}$$

Guess: $y_p = Ax^2 + Bx + Cxe^{-2x}$

$$y_p' = 2Ax + B + Ce^{-2x} - 2Cxe^{-2x}$$

$$y_p'' = 2A - 2Ce^{-2x} - 2Ce^{-2x} + 4Cxe^{-2x}$$

Plugging this into $y'' + 2y' = 2x + 5 - e^{-2x}$ we get

$$\underbrace{2A - 4Ce^{-2x} + 4Cxe^{-2x}}_{y_p''} + \underbrace{4Ax + 2B + 2Ce^{-2x} - 4Cxe^{-2x}}_{2y_p'} = 2x + 5 - e^{-2x}$$

As in 2(c) we need the $Ax^2 + Bx$ term but also since e^{-2x} appears in y_h also we want the $x e^{-2x}$ term in y_p

Combining like terms on the left-hand side we get

$$4Ax + (2A+2B) - 2Ce^{-2x} = 2x + 5 - e^{-2x}$$

So,

$$\begin{cases} 4A & = 2 & \textcircled{1} \\ 2A+2B & = 5 & \textcircled{2} \\ -2C & = -1 & \textcircled{3} \end{cases}$$

① gives $A = \frac{1}{2}$.

② gives $B = \frac{5}{2} - A = \frac{5}{2} - \frac{1}{2} = 2$

③ gives $C = \frac{1}{2}$.

Thus, $y_p = \frac{1}{2}x^2 + 2x + \frac{1}{2}xe^{-2x}$

Step 3: The general solution to

$$y'' + 2y' = 2x + 5 - e^{-2x} \text{ is}$$

$$y = y_h + y_p = c_1 + c_2 e^{-2x} + \frac{1}{2}x^2 + 2x + \frac{1}{2}xe^{-2x}$$